To Graham Roberts
with best wishes
Claudio Barone
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# On the numerical evaluation of Cauchy transforms

Claudio Barone and Ezio Venturino

*Dipartimento di Matematica, Università di Catania, 95100 Catania, Italy*

In this paper we consider simple methods for the reconstruction of the Cauchy transform over a curve when an explicit parametrization of the latter is not provided. The methods consist of replacing the parametrization of the curve by piecewise polynomial interpolation followed by the use of Newton–Cotes type formulae for the integration. The order of convergence of the resulting quadrature is higher than would be expected on the basis of considerations involving just interpolation theory, provided that the Cauchy transform is evaluated at known nodes on the curve. These results allow the calculation of the Cauchy transform at other points with the same accuracy if this scheme is followed by an interpolatory formula of sufficiently high accuracy.

**Keywords:** Numerical integration, quadrature formulae, interpolation.

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## 1. Introduction

The problem of the evaluation of Laplace or Fourier transforms is very important in applied mathematics, and special numerical methods have been developed for it. A related problem, perhaps less well known, is that of reconstructing the function defined by the Cauchy principal value of another function. Many physical problems can be reformulated by means of Cauchy transforms. Of particular importance are the applications to mixed boundary value problems of mathematical physics, for example fluid and aerodynamics, fracture mechanics and the theory of porous media. These problems, once suitably reformulated, lead to singular integral equations possessing a dominant term, which is interpreted as a Cauchy principal value integral. The study of numerical algorithms for their computation has been of wide interest in the literature of recent years.

The numerical evaluation of integrals formulated on curves is of considerable interest, mainly because many two-dimensional problems of mathematical physics can be reformulated as boundary integral equations. Three-dimensional problems are also under investigation. The resulting integral equation is formulated over a surface and the latter can be discretized via a simplicial approximation. The resulting numerical scheme is seen to possess a higher order of convergence than expected a priori. For the details we refer the reader to the very recent result of

Chien [2]. Here we remark only that the reason for the faster rate lies in the error cancellations over pairs of symmetric triangles about a common vertex.

In this paper we address the direct problem. We extend the analysis recently carried out for the case of line integrals for smooth functions [1] by considering the case when the transformed function is obtained by evaluating a Cauchy principal value line integral. Once the parametrization $r(t)$ is known, such an integral formulated over a line in $\mathbb{R}^2$ is not very much different from an integral over the interval [0, 1], and can be easily reduced to this classical case. Things change, however, if the line is known only at some points. In such a situation, the results show that the replacement of the parametrization by a suitable interpolant followed by the use of Newton–Cotes formulae yield a method that converges at a higher rate than can be expected by basing considerations on interpolation theory only.

Mathematically speaking, then, we would like to approximate the function

$$G(s) = \int_0^1 g(r(t))(l(t) - l(s))^{-1}|r'(t)| \, \mathrm{d}t, \quad s \in [0, 1], \tag{1}$$

where $r(t)$ is a smooth parametrization of a line in the plane, and $l(s)$ denotes arc length. If $s$ is fixed, $r(s)$ represents a fixed point. The problem then is just the evaluation of the integral. Otherwise, for $s$ variable one wants the reconstruction of the whole function $G$. The basic assumption that we make here is that we do not know the explicit parametrization of the line, $r(t)$, but that it is given only at some points. The investigation is a preliminary study to construct a reliable method for the calculation of the transformed function.

## 2. Background

To clarify the problem, we discuss at first the evaluation of the integral $\int_\gamma f(r) \, \mathrm{d}s$. We assume that the curve $\gamma$ is known only at some points and thus an explicit parametrization $r(t)$ is not available. We thus replace the curve with a piecewise polynomial interpolation, of suitable order $p$, thus obtaining the parametrization $r_p(t)$. If we then consider the problem of evaluating the integral with no singularity,

$$\int_0^1 f(r(t))|r'(t)| \, \mathrm{d}t, \tag{2}$$

we at first subdivide the interval of integration into subintervals $[t_{j-1}, t_j]$, using equispaced breakpoints $t_j = jh, j = 1, \ldots, n$, $h = 1/n$. On each subinterval we use quadrature nodes $s_{kj}$ and weights $w_{kj}$,

$$\begin{aligned} s_{kj} &= t_{j-1} + \eta_k, \quad \eta_k = (k-1)/(q-1), \quad k = 1, \ldots, q, \\ w_{kj} &= h\nu_{kj}, \qquad\qquad\qquad\qquad\qquad\quad j = 1, \ldots, n. \end{aligned}$$

Alternatively, we could just speak of the nodes $x_{ij}$ defined in the following way:

$$x_{ij} = t_{j-1} + (i-1)h/(p-1)\,, \quad i = 1,\ldots,p, j = 1,\ldots,n\,.$$

We can thus replace the integral by the approximation

$$\sum_{j=1}^{n}\sum_{k=1}^{q} w_{kj} f(r_p(s_{kj}))|r'_p(s_{kj})|\,.$$

Let us define the following quantities:

$$P = p \text{ if } p \text{ is even}\,, \quad P = p+1 \text{ if } p \text{ is odd}\,, \quad \hat{p} = P+1\,;$$

similar definitions hold also for $Q$, $\hat{q}$. Also put $\mu = \min(P, Q)$. Then the general convergence result for this scheme can be stated as follows.

THEOREM 1 [1]

If we assume the curve $\gamma$ to have a parametrization $r \in C^{p+2}[0, 1]$, with $r'(t) \neq 0$, than, for $f \in C^{Q}[0, 1]$, the order of convergence for a composite method which uses a $q$-point Newton–Cotes quadrature rule and a $p$-order interpolatory formula for the parametrization of the line is $\mu$.

*Remark*

This result shows that, if $p = q$, the order is the same as if we used the exact derivative $|r'(t)|$ in the quadrature formula.

## 3. The numerical procedure

Let $r(t)$, $a \leqslant t \leqslant b$, be a smooth parametrization of the curve $\gamma$ in $\mathbb{R}^2$. The problem we want to address here is the evaluation of the line integral (1), where $g$ represents a given smooth function, defined on $\gamma$, by means of the formula

$$\begin{aligned} Q_N = &\sum_{j=1}^{n}\sum_{k=1}^{q} w_{kj}[g(r_p(s_{kj})) - g(r_p(s))][l_p(s_{kj}) - l_p(s)]^{-1}|r'_p(s_{kj})| \\ &+ g(r_p(s))\ln|1 - l_p(1)/l_p(s)|\,, \end{aligned}$$

where

$$l_p(t) = \int_0^t |r'_p(u)|\,\mathrm{d}u\,.$$

For the estimation of the error, we observe that

$$\begin{aligned} G(s) = &\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} [g(r(t)) - g(r(s))][l(t) - l(s)]^{-1}|r'(t)|\,\mathrm{d}t \\ &+ g(r(s))\ln|1 - l(1)/l(s)|\,. \end{aligned}$$

Let us now define the functions

$$f(r(t)) = [g(r(t)) - g(r(s))][l(t) - l(s)]^{-1},$$

$$f(r_p(t)) = [g(r_p(t)) - g(r_p(s))][l_p(t) - l_p(s)]^{-1}.$$

For studying the error, we can concentrate on one interval; without loss of generality, we can take it to be $[0, h]$. Notice that $s$ may or may not belong to it. Locally then, the error can be rewritten as

$$\int_0^h f(r(t))|r'(t)|\, \mathrm{d}t - \sum_{k=1}^{q} w_{k1} f(r_p(s_{k1}))|r'_p(s_{k1})| = A + B + C,$$

where

$$A \equiv \int_0^h [f(r(t)) - f(r_p(t))]|r'_p(t)|\, \mathrm{d}t,$$

$$B \equiv \int_0^h f(r_p(t))[|r'(t)| - |r'_p(t)|]\, \mathrm{d}t,$$

$$C \equiv \int_0^h f(r_p(t))|r'_p(t)|\, \mathrm{d}t - h\sum_{k=1}^{q} \nu_{k1} f(r_p(s_{k1}))|r'_p(s_{k1})|.$$

The key fact to be observed here is stated in the following

LEMMA

Suppose $s \in [0, h]$. Then $g \in C^{Q+1}[0, h]$ if and only if $f \in C^{Q}[0, h]$.

In order to apply theorem 1, we require $g \in C^{\hat{q}}[0, 1]$, which is a stronger assumption than the one necessary for the evaluation of integrals not containing the Cauchy principal value singularity. The estimates for each term $A, B, C$ defined above now follow from the results of [1]. Since in view of the above discussion we have the required smoothness for the integrand function $f$, local convergence is one order higher than given in theorem 1. We thus obtain

$$A = O(h^{\hat{p}}), \quad B = O(h^{\hat{p}}), \quad C = O(h^{\hat{q}}).$$

Thus, locally the overall error is $O(h^{\mu+1})$. Globally, one order will be lost, since we are summing over $n$ such intervals. Before stating our convergence result, however, we need to take into account an extra error term. This is given by

$$g(r(s)) \ln|1 - l(1)/l(s)| - g(r_p(s)) \ln|1 - l_p(1)/l_p(s)| = D + E,$$

with

$$D \equiv g(r(s))[\ln|1 - l(1)/l(s)| - \ln|1 - l_p(1)/l_p(s)|],$$

$$E \equiv [g(r(s)) - g(r_p(s))] \ln|1 - l_p(1)/l_p(s)|.$$

For the term $D$, we observe that $g(r(s))$ is constant; using theorem 1 with $f \equiv 1$, it follows that for $0 \leqslant s \leqslant 1$

$$l_p(s) = \int_0^s |r_p'(t)| \, \mathrm{d}t = l(s) + O(h^{\mu}) .$$

For the term in braces, using Taylor's expansion,

$$[\ln |1 - l(1)/l(s)| - \ln |1 - l(1)/l_p(s)|]$$
$$= \ln |1 - l(s)/l(1)| - \ln |1 - l(s)/l_p(1)| + \ln |l_p(s)/l(s)| - \ln |l_p(1)/l(1)| = O(h^{\mu}) .$$

For the term $E$ we have, using again the approximation for $l_p(s)$ found above,

$$\ln |1 - l_p(1)/l_p(s)| = constant + O(h^{\mu})$$

Table 1
Midpoint rule.

| Time | $n$ | Value | Difference | Order |
|---|---|---|---|---|
| $r(t) = (\cos t, \sin t)$, $0 \leqslant t \leqslant 0.83$, $p = 3$ | | | | |
| 0.06 | 1 | −0.05140089384484 | | |
| 0.00 | 2 | −0.05442121232860 | −0.30203E − 02 | |
| 0.00 | 4 | −0.05506358602527 | −0.64237E − 03 | 2.23 |
| 0.11 | 8 | −0.05520696451904 | −0.14338E − 03 | 2.16 |
| 0.27 | 16 | −0.05524071392932 | −0.33749E − 04 | 2.09 |
| 0.94 | 32 | −0.05524902770245 | −0.83138E − 05 | 2.02 |
| 3.29 | 64 | −0.05525100214483 | −0.19744E − 05 | 2.07 |
| 12.47 | 128 | −0.05525149325916 | −0.49111E − 06 | 2.01 |
| $r(t) = (\cos t, \sin t)$, $0 \leqslant t \leqslant 4.3$, $p = 4$ | | | | |
| Time | $n$ | Value | Difference | Order |
| 0.00 | 1 | 1.86098079934359 | | |
| 0.00 | 2 | 2.52380055492909 | 0.66282E+00 | |
| 0.06 | 4 | 2.28100303074699 | −0.24280E + 00 | 1.45 |
| 0.16 | 8 | 2.23606137498578 | −0.44942E − 01 | 2.43 |
| 0.44 | 16 | 2.22589460449729 | −0.10167E − 01 | 2.14 |
| 1.27 | 32 | 2.22339950398907 | −0.24951E − 02 | 2.03 |
| 4.39 | 64 | 2.22277834482288 | −0.62116E − 03 | 2.01 |
| 16.48 | 128 | 2.22262321575953 | −0.15513E − 03 | 2.00 |
| $r(t) = (\cos t, \sin t)$, $0 \leqslant t \leqslant 6.1$, $p = 5$ | | | | |
| Time | $n$ | Value | Difference | Order |
| 0.00 | 1 | 5.06877288024913 | | |
| 0.06 | 2 | 2.71645434140859 | −0.23523E + 01 | |
| 0.05 | 4 | 2.95533745208494 | 0.23888E+00 | 3.30 |
| 0.22 | 8 | 2.90435154772082 | −0.50986E − 01 | 2.23 |
| 0.61 | 16 | 2.89797199721279 | −0.63796E − 02 | 3.00 |
| 2.14 | 32 | 2.89630808706733 | −0.16639E − 02 | 1.94 |
| 7.69 | 64 | 2.89588810096551 | −0.41999E − 03 | 1.99 |
| 28.94 | 128 | 2.89578286227432 | −0.10524E − 03 | 2.00 |

Table 2
Trapezoidal rule.

| Time | $n$ | Value | Difference | Order |
|---|---|---|---|---|
| $r(t) = (3\cos t, 2\sin t)$, $0 \leqslant t \leqslant 1$, $p = 3$ | | | | |
| 0.05 | 1 | 0.00779846699033 | | |
| 0.00 | 2 | 0.00685446893691 | $-0.94400\text{E}-03$ | |
| 0.11 | 4 | 0.00660592214600 | $-0.24855\text{E}-03$ | 1.93 |
| 0.11 | 8 | 0.00653966901771 | $-0.66253\text{E}-04$ | 1.91 |
| 0.27 | 16 | 0.00652656005375 | $-0.13109\text{E}-04$ | 2.34 |
| 0.99 | 32 | 0.00652417630102 | $-0.23838\text{E}-05$ | 2.46 |
| 3.30 | 64 | 0.00652358930531 | $-0.58700\text{E}-06$ | 2.02 |
| 12.25 | 128 | 0.00652345570581 | $-0.13360\text{E}-06$ | 2.14 |
| $r(t) = (3\cos t, 2\sin t)$, $0 \leqslant t \leqslant 2$, $p = 4$ | | | | |
| Time | $n$ | Value | Difference | Order |
| 0.00 | 1 | 0.45556338841095 | | |
| 0.05 | 2 | 0.25631549052933 | $-0.19925\text{E}+00$ | |
| 0.06 | 4 | 0.18896032050807 | $-0.67355\text{E}-01$ | 1.56 |
| 0.16 | 8 | 0.17110107639927 | $-0.17859\text{E}-01$ | 1.92 |
| 0.50 | 16 | 0.16653760357916 | $-0.45635\text{E}-02$ | 1.97 |
| 1.48 | 32 | 0.16539015108600 | $-0.11475\text{E}-02$ | 1.99 |
| 4.83 | 64 | 0.16510237834210 | $-0.28777\text{E}-03$ | 2.00 |
| 17.52 | 128 | 0.16503041626266 | $-0.71962\text{E}-04$ | 2.00 |
| $r(t) = (3\cos t, 2\sin t)$, $0 \leqslant t \leqslant 3$, $p = 5$ | | | | |
| Time | $n$ | Value | Difference | Order |
| 0.05 | 1 | 8.60612004080361 | | |
| 0.06 | 2 | 4.06680504284273 | $-0.45393\text{E}+01$ | |
| 0.11 | 4 | 2.67878926412375 | $-0.13880\text{E}+01$ | 1.71 |
| 0.33 | 8 | 2.31666238797912 | $-0.36213\text{E}+00$ | 1.94 |
| 0.82 | 16 | 2.22659960314194 | $-0.90063\text{E}-01$ | 2.01 |
| 2.69 | 32 | 2.20410381209595 | $-0.22496\text{E}-01$ | 2.00 |
| 9.18 | 64 | 2.19847710763292 | $-0.56267\text{E}-02$ | 2.00 |
| 33.88 | 128 | 2.19706942113089 | $-0.14077\text{E}-02$ | 2.00 |

and

$$[g(r(s)) - g(r_p(s))] = -\nabla g(r(s))[r_p(s) - r(s)] + o[r_p(s) - r(s)] = O(h^p) .$$

Thus, overall $E = O(h^p)$. We have thus shown the following

THEOREM 2

For the singular integral, if the hypotheses of theorem 1 on the parametrization of the curve are satisfied, and if $g \in C^{\hat{q}}[0, 1]$, then for a composite method which uses a $q$-point Newton–Cotes quadrature rule and a $p$-order interpolatory formula for the parametrization of the line, the convergence rate is given by $p$.

Table 3
Simpson's rule.

| $r(t)=(t^3,t^2)$, $0\leqslant t\leqslant 1$, $p=4$ | | | | |
|---|---|---|---|---|
| Time | $n$ | Value | Difference | Order |
| 0.05 | 1 | −1.00672363805871 | | |
| 0.06 | 2 | −1.05193380160156 | −0.45210E − 01 | |
| 0.16 | 4 | −1.05286844725913 | −0.93465E − 03 | 5.60 |
| 0.44 | 8 | −1.05296884115155 | −0.10039E − 03 | 3.22 |
| 1.21 | 16 | −1.05297391067503 | −0.50695E − 05 | 4.31 |
| 4.07 | 32 | −1.05297403372308 | −0.12305E − 06 | 5.36 |
| 14.83 | 64 | −1.05297404230799 | −0.85849E − 08 | 3.84 |
| 55.75 | 128 | −1.05297404342723 | −0.11192E − 08 | 2.94 |
| $r(t)=(3\cos t, 2\sin t)$, $0\leqslant t\leqslant 3$, $p=5$ | | | | |
| Time | $n$ | Value | Difference | Order |
| 0.06 | 1 | 2.06378045870071 | | |
| 0.11 | 2 | 1.63176432005606 | −0.43202E + 00 | |
| 0.21 | 4 | 1.66460651464199 | 0.32842E − 01 | 3.72 |
| 0.66 | 8 | 1.65708743921516 | −0.75191E − 02 | 2.13 |
| 1.93 | 16 | 1.65674192919427 | −0.34551E − 03 | 4.44 |
| 6.42 | 32 | 1.65671835978824 | −0.23569E − 04 | 3.87 |
| 22.74 | 64 | 1.65671724388236 | −0.11159E − 05 | 4.40 |
| 86.29 | 128 | 1.65671708739539 | −0.15649E − 06 | 2.83 |

*Remarks*

(1) If $s$ is known but $r(s)$ is not, and if $p<q$, the result of theorem 1 does not carry over to this more general case. The error will be $O(h^p)$ and not, as for the ordinary integral, $O(h^P)$. Notice, however, that for $p$ even, the two errors coincide.

(2) If $r(s)$ is a *known* node, then $r_p(s)\equiv r(s)$, so that $g(r_p(s))\equiv g(r(s))$. Then part $E$ of the error disappears and the overall error as for the ordinary integral will then be $O(h^P)$.

To reconstruct the function $G(s)$, these remarks show that the collocation points at which to evaluate the transform, after the integral is discretized via the quadrature formula, are best taken to be the known nodes on the curve. From these $G(x_{ij})$, the function can be reconstructed by means of piecewise polynomial interpolation of order greater than or equal to $\mu$. In summary,

THEOREM 3

Under the hypotheses of the previous theorems, if the integrand $g\in C^{\hat{q}}[0,1]$, the reconstruction of the Cauchy transform $G(s)$ can be obtained with accuracy $O(h^\mu)$ on the basis of Newton–Cotes formulae and interpolation over the curve $\gamma$, provided that we select the collocation points to be the known points of the curve $\gamma$, and if $G$ is needed at other points, provided that the quadrature scheme is followed by a suitably high interpolation formula.

## 4. Conclusions

To support our analysis, in this section we present some numerical evidence. In tables 1–3, we give some results for various cases involving low order Newton–Cotes formulae up to order 3 and interpolation of order up to 5. All the computations have been performed on a 386-based machine, in double precision accuracy. In all the examples the integrand is $f(x, y) = \exp[-(x + y)]$. The position of the singularity is always at $s = 0.8$. The parametrization of the curve and the range for the parameter is indicated in each case.

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